Outline

- Review of community detection
- Community extraction
- Simulation study
- Real data analysis
- Asymptotic consistency
- Future work

Network data

Data: links between nodes

- Social and friendship networks, citation networks
- Marketing, recommender systems
- Computer, mobile, sensor networks
- World Wide Web
- Gene regulatory networks, food webs

Notation

Given a network $N = (V, E)$

- V is the set of nodes, E is the set of edges.
- N is represented by its adjacency matrix A:

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge from node } i \text{ to node } j, \\ 0 & \text{otherwise.} \end{cases}$$

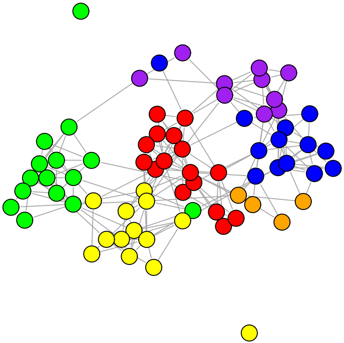
- A can be symmetric (undirected network) or asymmetric (directed network).

Community detection

- Communities: many links within and few links between
- Community detection is typically formulated as finding a partition $V = V_1 \cup \cdots \cup V_K$ which gives “tight” communities in some suitable sense.
- For simplicity, give criteria for partitioning into two communities V_1 and V_2.

Example: a school friendship network

Colors represent grades



Graph cuts

- Min-cut: minimize

$$R = \sum_{i \in V_1, j \in V_2} A_{ij} \ .$$

Trivial solution of $V_1 = V$ or $V_2 = V$.

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- Ratio cut (Wei and Cheng, 1989): minimize

$$\frac{R}{|V_1| \cdot |V_2|},$$

  where $|V_1|$ and $|V_2|$ are the sizes of the two communities.

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- Ratio cut (Wei and Cheng, 1989): minimize
$$\frac{R}{|V_1| \cdot |V_2|},$$
where $|V_1|$ and $|V_2|$ are the sizes of the two communities.
- Normalized cut (Shi and Malik, 2000): minimize
$$\frac{R}{D_1} + \frac{R}{D_2},$$
where $D_k = \sum_{i \in V_k, j \in V} A_{ij}$ is the total number of edges from nodes in V_k.

$$Q = \sum_k \left[\frac{O_{kk}}{L} - \left(\frac{D_k}{L} \right)^2 \right]$$

- Q is the sum of observed - expected under the configuration model: probability of edge between nodes with degrees d_i, d_j is $d_i d_j / L$.
- Typically solved by an eigenvalue method via relaxing $\max_{s_i = \pm 1} \boldsymbol{s}^\top \boldsymbol{M} \boldsymbol{s}$ to $\max_{\|\boldsymbol{s}\| = 1} \boldsymbol{s}^\top \boldsymbol{M} \boldsymbol{s}$.

Limitation of partition methods

- Many real-world networks contain nodes with few links that may not belong to any community (“background”).
- The “strength” of a community depends on links between nodes not related to the community.
- Determining the number of communities is difficult.

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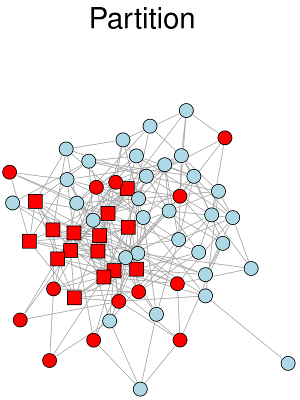
Community extraction

- Allow for background nodes that only have sparse links to other nodes.
- Extract communities sequentially: at each step look for a set with a large number of links within and a small number of links to the rest of the network.
- Stop when no more meaningful communities exist.

Toy example

- One community with 15 nodes, total 60 nodes.
- Links between community members form independently with probability 0.5.
- Links between community members and other nodes form independently with probability 0.1.
- Links between other nodes form independently with probability 0.1.
- Compare partition into two communities (via modularity) to extraction of a single community.

Shapes represent the truth, colors represent results.



Extraction criterion

Maximize

$$W(S) = \frac{O(S)}{|S|^2} - \frac{B(S)}{|S| \cdot |S^c|} ,$$

where

$$O(S) = \sum_{i,j \in S} A_{ij} , \; B(S) = \sum_{i \in S, j \in S^c} A_{ij} .$$

The links within the complement of set S do not matter.

Adjusted criterion

- In sparse networks, tends to pick small disconnected components first.
- To avoid small communities, can use

Maximize

$$W_a(S) = |S| \cdot |S^c| \left(\frac{O(S)}{|S|^2} - \frac{B(S)}{|S| \cdot |S^c|} \right) .$$

The factor $|S| \cdot |S^c|$ encourages more balanced solutions.

Algorithm

- Tabu Search (Glover, 1986; Glover and Laguna, 1997): a local optimization technique based on label switching.
- Switch labels to improve the value of the criterion but each node has to keep its label for at least T iterations.
- Run the algorithm for many randomly ordered nodes.

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Numerical evaluation

- S is the extracted community.
- C_S is the true community that matches S best.

PPV and NPV

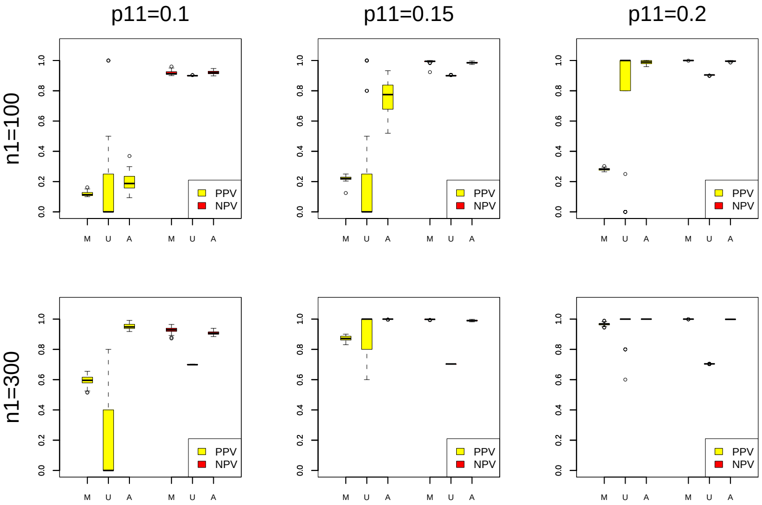
$$\text{PPV} = \frac{|C_S \cap S|}{|S|} \qquad \text{Purity}$$

$$\text{NPV} = 1 - \frac{|C_S \cap S^c|}{|S^c|} \qquad \text{Completeness}$$

Simulation I

- One community with background
- $n = 1000$
- $n_1 = 100, 200, 300$
- $p_{12} = 0.05$, $p_{22} = 0.05$
- $p_{11} = 0.1, 0.15, 0.2$

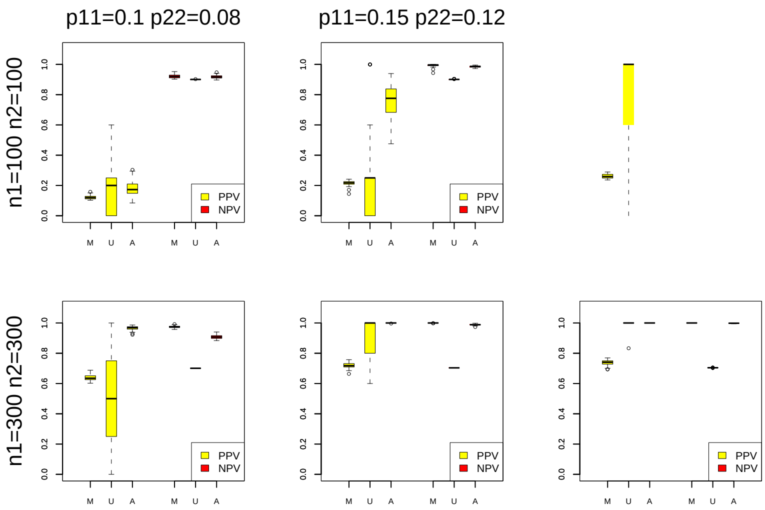
Results of simulation I



Simulation II

- Two communities plus background
- $n = 1000$
- $n_1 = 100, 300, n_2 = 100, 300$
- $p_{12} = p_{23} = p_{13} = p_{33} = 0.05$
- $p_{11} = 0.1, 0.15, 0.2$
- $p_{22} = 0.08, 0.12, 0.16$

Results for simulation II



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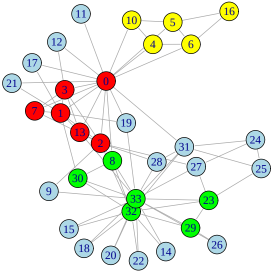
Karate club network

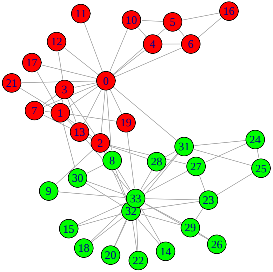
- Friendships between 34 members of a karate club (Zachary, 1977).
- This club has subsequently split into two parts following a disagreement between an instructor (node 0) and an administrator (node 33).

Karate club network

Community extraction



Modularity



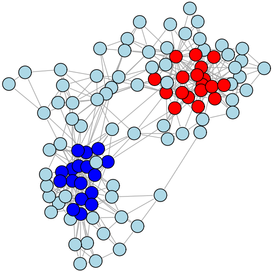
Political books network

Links in the political books network (Newman, 2006) represent pairs of books frequently bought together on amazon.com.

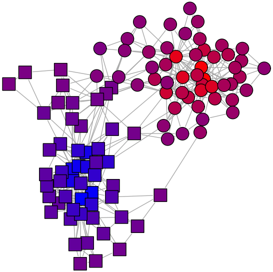
Blue: liberal
Red: conservative

Political books network

Community extraction



Modularity

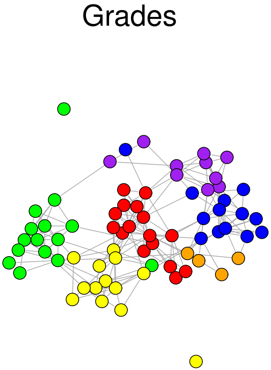


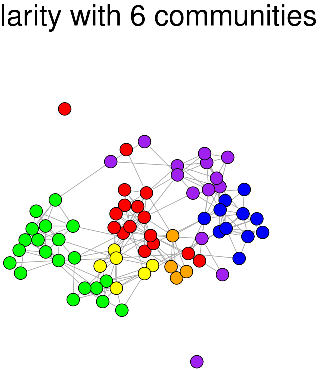
School friendship network

The school friendship network is complied from the National Longitudinal Study of Adolescent Health (AddHealth).

Grade 7: red
Grade 8: blue
Grade 9: green
Grade 10: yellow
Grade 11: purple
Grade 12: orange

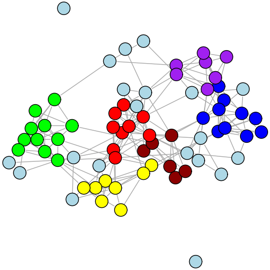
School friendship network



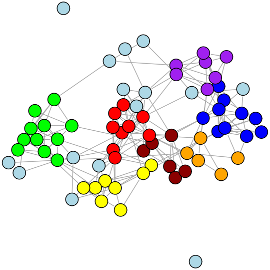


School friendship network

Extracting 6 communities



Extracting 7 communities



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Block models

One of the simplest random graph models for communities

- Each node is assigned to a block independently of other nodes, with probability π_k for block k, $\sum_{k=1}^{K} \pi_k = 1$.
- Given that node i belongs to block a and node j belongs to block b, $P[A_{ij} = 1] = p_{ab}$, and all edges are independent.
- Parametrized as $P_n = \rho_n P$, where $\rho_n = P_n[A_{ij} = 1] \to 0$.
- Expected node degree $\lambda_n = n\rho_n$
- Can stipulate background: assume $p_{aK} < p_{bb}$ for all $a = 1, \ldots, K$, and all $b = 1, \ldots, K - 1$.

Asymptotic consistency result

- For simplicity, assume one community and background ($K = 2$ with parameters $p_{11}, p_{12}, p_{22}, \pi$).
- Let $\boldsymbol{c}$ be the true labels, $\hat{\boldsymbol{c}}^{(n)}$ the estimated labels.

Theorem

For any $0 < \pi < 1$, *if* $p_{11} > p_{12}$, $p_{11} > p_{22}$ *and* $p_{11} + p_{22} > 2p_{12}$, $\frac{\lambda_n}{\log n} \to \infty$, *the maximizer* $\hat{\boldsymbol{c}}^{(n)}$ *of both unadjusted and adjusted criteria satisfies*

$$P[\hat{\boldsymbol{c}}^{(n)} = \boldsymbol{c}] \to 1 \quad as \quad n \to \infty.$$

- Holds for $p_{12} = p_{22} = p < p_{11}$
- Proof: apply Bickel and Chen (PNAS, 2009)

Bickel & Chen consistency framework

- Assume a block model with known K
- Given a proposed label assignment $\boldsymbol{s}$, true labels $\boldsymbol{c}$, let R be the confusion matrix with

$$R_{ab}(\boldsymbol{s},\boldsymbol{c}) = \frac{1}{n}\sum_{i=1}^{n} I(s_i = a, c_i = b) \ .$$

- Many criteria, including ours, can be written as a function of the confusion matrix.
- Key condition: the population version of the criterion is maximized by the “correct” confusion matrix $diag(\pi_1, \ldots, \pi_k)$.

Future work

- Eigenvalue method
- Determining the number of communities
- Adjusted criterion

$$W_a(S) = \left(|S| \cdot |S^c|\right)^{\alpha} \left(\frac{O(S)}{|S|^2} - \frac{B(S)}{|S| \cdot |S^c|}\right)$$